

Analytical form of light-ray tracing in invisibility cloaks

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Abstract

In this paper, we review the methodology of transformation optics, which can construct invisibility cloak through the transformation of coordinates based on the form invariance of Maxwell's equations. Three different ways to define the components of electromagnetic fields are compared for removing some ambiguities. The analytical expressions of light-ray and wave-normal ray are derived in spherical and cylindrical ideal invisibility cloaks created with any continuous radial transformation functions, and their physical interpretation is also given. Using the duality principle in anisotropic media, we prove that light-ray vector satisfies “ray-vector eikonal equation” corresponding to the usual “wave-vector eikonal equation”. The results interpret why the wave vector maps to the ray vector transferring from the virtual space to the physical space, but not the wave vector. As an application, we investigate the special transformation functions which make the light-ray function satisfy harmonic equation.

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I. INTRODUCTION

Recently, invisibility cloak has attracted much attention and widely study [1–19]. Its theory is based on the form invariance of 3-dimensional Maxwell’s equations under a coordinate transformation [1, 4–6]. Briefly, the property of material, such as ε , μ in a vacuum Euclidean space with an original orthogonal coordinate, will be transformed into a different form through a coordinate transformation. However, the form of Maxwell’s equations is invariant. The transformed ε and μ could be alternatively interpreted as a new different set of material properties in the original orthogonal coordinate system. Based on this point, the developments have generated a new subject of transformation optics [7–9]. Meanwhile, a conformal mapping method has been applied to design a medium which can achieve invisibility in the geometric limit [2]. The validation of the invisible cloak designed with transformation optics method has been verified through several ways, including ray tracing approach [4, 17], full-wave simulations [10], and analysis based on Mie scattering [11–16]. The early studies in the invisibility cloak were mostly based on linear transformation proposed by Pendry *et al.* [1, 4, 11–13]. Then different transformations have also been discussed [14, 16]. Luo *et al.* proved that the permittivity and permeability created with a most general form of transformation can realize spherical invisibility cloaks and give the analytical calculation of electromagnetic fields for any spherical cloak using Mie scattering method [15].

In the pioneering work of Pendry *et al.*[1], it had been noted that the Poynting vectors acting as the tangent vectors of light should propagate to surround the hidden area in the cloak layer. They used the canonical equations of a double anisotropic material optical system to describe the light propagation [4]. According to the analogy between the curved space and double anisotropic media, Ref. [17] described the ray traces with the geodesic equation in the virtual curved system, and gave the proof that it is equivalent to canonical equations. Since the original Euclidean space (virtual space) is a vacuum, the Poynting vectors and wave vectors of a beam of light traveling in it are parallel to each other. Whereas, when mapping to physical space where a cloak is placed, Poynting vectors and wave vectors are not paralleling any more, which has been pointed out and discussed by Leonhardt in [20]. In this paper, however, we will expound this point from a different perspective by using “ray-vector eikonal equation” and corresponding relations.

The paper is organized as follows. At the beginning, we compare three common ways to define the components of electromagnetic fields in 3-D curvilinear coordinates, which are named Minkowski’s definition, Landau’s definition, and the components in normalized bases. To do this is benefit to clarify some ambiguities caused by the mistiness of definition. After that, We repeat the fundamentals of transformation optics, which contain the corresponding relations between the curvilinear coordinate system of virtual space (**S** system) and real

physical space (\mathbf{P} space), and emphasize that the validity of transformation optics is not only based on the form of Maxwell's equations, but also based on the correspondence of boundary conditions between the two systems.

After explication of the fundamentals, the analytical solution of electromagnetic fields is derived in spherical and cylindrical cloaks with arbitrary radial transformation functions. The Poynting vectors and wave vectors are further calculated, and the analytical expressions of light rays and wave-normal rays are obtained, respectively. Then, we apply the expressions to four types of cloak made by four different transformation functions. Furthermore, we provide the physical interpretation to the analytical solution of light-rays, and verify it satisfies the geodesic equation in \mathbf{S} system discussed in [17]. Through duality principle of anisotropic material, we obtain "ray-vector eikonal equation" corresponding to the usual "wave-vector eikonal equation". And through the corresponding relations, we prove that the covariant component of wave vector k_i in \mathbf{S} system maps to wave vector $k_{(\mathbf{P})i}$ in \mathbf{P} space, however, its contravariant component k^i maps to ray vector $s_{(\mathbf{P})}^i$ in \mathbf{P} space. This is the precise reason that wave vector and Poynting vector are split in \mathbf{P} space.

Finally, as an application, we find out the general form of transformation functions that makes the light-ray expression satisfy harmonic equation. It is just the conformal transformation function, which Leonhardt applied to design invisibility device in the geometric limit [2]. In this case, Poynting vectors and wave vectors have neither divergence nor curl.

II. COORDINATE TRANSFORMATION OF ELECTROMAGNETIC QUANTITIES

A. Definition of electromagnetic fields in 3-D orthogonal curvilinear coordinates

For an arbitrary 3-D curvilinear coordinate system $\{x^i\}$ to describe spatial part of a flat spacetime, its bases $\vec{e}_i = \partial\vec{r}/\partial x^i$ are often called coordinate bases or holonomic bases [21], and its spatial metric is $\gamma_{ij} = \vec{e}_i \cdot \vec{e}_j$. The components of a tensor $\vec{\vec{T}}$ are defined by $\vec{\vec{T}} = T^{ij}\vec{e}_i\vec{e}_j$.

The component forms of Maxwell's equations in any 3-D curvilinear coordinate system are:

$$\nabla_i \bar{B}^i = \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \bar{B}^i) = 0, \quad \frac{\partial \bar{B}^i}{\partial t} + \epsilon^{ijk} \partial_j \bar{E}_k = \frac{\partial \bar{B}^i}{\partial t} + \frac{1}{\sqrt{\gamma}} e^{ijk} \partial_j \bar{E}_k = 0, \quad (1a)$$

$$\nabla_i \bar{D}^i = \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \bar{D}^i) = \rho, \quad -\frac{\partial \bar{D}^i}{\partial t} + \epsilon^{ijk} \partial_j \bar{H}_k = -\frac{\partial \bar{D}^i}{\partial t} + \frac{1}{\sqrt{\gamma}} e^{ijk} \partial_j \bar{H}_k = 0, \quad (1b)$$

where ∇_i is 3-D covariant derivative, γ is the determinate of γ_{ij} , e^{ijk} is 3-order completely antisymmetric symbol, and $\epsilon^{ijk} = e^{ijk}/\sqrt{\gamma}$ is 3-order Levi-Civita tensor. We use Latin indices

i, j, \dots to denote 1 to 3 for three spatial components. The components of electromagnetic vectors in these equations are resolved in holonomic bases, and this mode of components is named Landau's definition (see Appendix A for general Landau's definition in 4-D curved spacetime). We add a bar on these components to discriminate Minkowski's definition, which will be introduced below.

In this representation, \vec{E} , \vec{H} , \vec{D} , \vec{B} are all treated as 3-D spatial vectors, which imply their components satisfy the law of vector transformation $T^i = (\partial x^i / \partial x'^{k'}) T'^{k'}$, meanwhile $\vec{\varepsilon}$ and $\vec{\mu}$ are treated as 3-D spatial tensors, and their components satisfy the law of transformation $\bar{\varepsilon}^{ij} = \Lambda_{i'}^i \Lambda_{j'}^j \bar{\varepsilon}'^{i'j'}$, $\bar{\mu}^{ij} = \Lambda_{i'}^i \Lambda_{j'}^j \bar{\mu}'^{i'j'}$, where $\Lambda_{i'}^i = \partial x^i / \partial x'^{i'}$ is the element of Jacobian matrix. In vacuum, $\bar{\varepsilon}^{ij} / \varepsilon_0 = \bar{\mu}^{ij} / \mu_0 = \gamma^{ij}$.

In transformation optics, a different way to define electromagnetic fields in curved coordinate is established in order to hold the expressions of Maxwell's equations in Cartesian coordinate [6, 7, 22]. If we define electromagnetic quantities as

$$D^i = \sqrt{\gamma} \bar{D}^i, \quad B^i = \sqrt{\gamma} \bar{B}^i, \quad (2a)$$

$$\varepsilon^{ij} = \sqrt{\gamma} \bar{\varepsilon}^{ij}, \quad \mu^{ij} = \sqrt{\gamma} \bar{\mu}^{ij}. \quad (2b)$$

$$\hat{\rho} = \sqrt{\gamma} \rho, \quad \hat{j}^i = \sqrt{\gamma} j^i, \quad (2c)$$

substituting them into Eqs. (1), the form of Maxwell's equations in curved coordinates changes to the form appearing in Cartesian coordinates. As a result, the form of Maxwell's equations expressed by ordinary derivative is invariant from coordinate transformation. We name this mode of definition as Minkowski's definition (see Appendix A for general Minkowski's definitions).

Note that the above components D^i , B^i , ε^{ij} , μ^{ij} are not the real components of the corresponding vectors or tensors in the curvilinear coordinate, but they can be regarded as the components of $\sqrt{\gamma} \vec{D}$, $\sqrt{\gamma} \vec{B}$, $\sqrt{\gamma} \vec{\varepsilon}$, $\sqrt{\gamma} \vec{\mu}$ respectively. These quantities are all vector or tensor density, which satisfy the law of transformation [4]

$$D^i = \left| \det (\Lambda_{i'}^i)^{-1} \right| \Lambda_{i'}^i D'^{i'}, \quad B^i = \left| \det (\Lambda_{i'}^i)^{-1} \right| \Lambda_{i'}^i B'^{i'}, \quad (3a)$$

$$\varepsilon^{ij} = \left| \det (\Lambda_{i'}^i)^{-1} \right| \Lambda_{i'}^i \Lambda_{j'}^j \varepsilon'^{i'j'}, \quad \mu^{ij} = \left| \det (\Lambda_{i'}^i)^{-1} \right| \Lambda_{i'}^i \Lambda_{j'}^j \mu'^{i'j'}. \quad (3b)$$

In vacuum, the permittivity and permeability have the simple forms as

$$\varepsilon^{ij} = \varepsilon_0 \sqrt{\gamma} \gamma^{ij}, \quad \mu^{ij} = \mu_0 \sqrt{\gamma} \gamma^{ij}. \quad (4)$$

The components resolved by unit basis are also commonly used to expressing tensors in curved coordinates. The set of normalized bases is defined as $\hat{e}_i = \vec{e}_i / \|\vec{e}_i\| = \vec{e}_i / \sqrt{\gamma_{ii}}$, yet these bases are anholonomic in that we usually can not find a set of coordinates to satisfy $\hat{e}_i =$

$\partial\vec{r}/\partial x^i$. The components in orthonormal bases are also defined by $\vec{T} = T_{\langle ij \rangle} \hat{e}_i \hat{e}_j$, and there is no distinction between covariant and contravariant components when the coordinates are orthogonal. For a 3-D orthogonal curvilinear coordinate system the set of orthonormal bases is corresponding to the tetrad in 4-D spacetime (see Appendix A). Since all the components of a vector in unit bases have the same dimension, they act as the measurement of the vector in each direction. In terms of the relation of holonomic and anholonomic bases, we can get

$$\varepsilon^{\langle ij \rangle} = h_i h_j \bar{\varepsilon}^{ij} = \frac{h_i h_j}{\sqrt{\gamma}} \varepsilon^{ij}, \quad \mu^{\langle ij \rangle} = h_i h_j \bar{\mu}^{ij} = \frac{h_i h_j}{\sqrt{\gamma}} \mu^{ij}, \quad (5)$$

where $h_i = \sqrt{\gamma_{ii}}$. For orthogonal coordinates, $\sqrt{\gamma} = h_1 h_2 h_3$. In Eqs. (5), we have canceled the Einstein summation convention. For vacuum, $\varepsilon_{\langle ij \rangle}/\varepsilon_0 = \mu_{\langle ij \rangle}/\mu_0 = \delta_{ij}$.

B. Three coordinate systems in transformation optics

The method of transformation optics starts from a virtual 3-D flat Euclidean space with medium of vacuum. An orthogonal coordinate system (Cartesian or curvilinear coordinate), \mathbf{S}' system, is employed to describe the virtual space. In \mathbf{S}' , the coordinates are written as $x'^{i'}$ and the metric is $\gamma'_{i'j'}$. However, another coordinate system \mathbf{S} with coordinates x^i and metric γ_{ij} can be established to describe it. The transformation of coordinates is $x^i = x^i(x'^{i'})$.

If we investigate electromagnetic fields propagating in the virtual space, Eqs. (4) give the Minkowski's form of permittivity and permeability describing vacuum either in \mathbf{S} or in \mathbf{S}' . And Eqs. (3) present the transformation law of Minkowski's form between \mathbf{S} and \mathbf{S}' system.

Due to the invariance of Maxwell's equations under Minkowski's definition, we can alternatively interpret the Minkowski's form of permittivity and permeability in \mathbf{S} system as the properties of a real material in physical space (\mathbf{P} space). The \mathbf{P} space is also an Euclidean space and described by the same orthogonal coordinates as \mathbf{S}' . We use $x_{(\mathbf{P})}^i$ to denote the coordinates of \mathbf{P} , and also add the subscript “(P)” to other physical quantities in \mathbf{P} space for definitude. Since we employ the same coordinate system as \mathbf{S}' system, the spatial metric $\gamma_{(\mathbf{P})ij}$ of \mathbf{P} space hold an identical form with $\gamma'_{i'j'}$. On the other hand, the Minkowski's form of electromagnetic quantities in \mathbf{P} space takes the same formula as that in \mathbf{S} system of virtual space (see Fig. 1).

In previous articles, the \mathbf{S} system of virtual space is often called a curved space because of its nontrivial metric. In mathematics, it is certainly a metric space or manifold. However, considering its zero curvature, we only treat it as a curvilinear coordinate system of virtual flat space. In this paper, we only call a system space when they have a real physical background of spacetime, such as virtual space and \mathbf{P} space. But for the coordinate systems

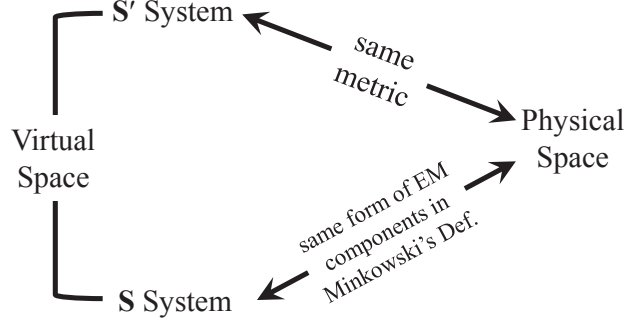


FIG. 1. Relations among Physical space, \mathbf{S}' coordinate system and \mathbf{S} coordinate system of virtual space.

used to describe a space, we only call them system but not space. Nevertheless, we do not distinguish the \mathbf{P} space itself and its coordinate system, because we only set one system to describe it.

The corresponding relations of electromagnetic quantities between physical space and \mathbf{S} system are shown in Table I. There are several points to emphasize. First, corresponding means the Minkowski's components of these quantities regarded as functions of coordinates have same mathematical forms in the two systems. Second, for an individual quantity, only the covariant (or contravariant) components hold the same form in the two systems, yet the other kind of components isn't congruent. For instance, $D_{(\mathbf{P})}^i(x_{(\mathbf{P})}^i)$ and $D^i(x^i)$ have the same mathematical form, but the forms of $D_{(\mathbf{P})i}(x_{(\mathbf{P})}^i)$ and $D_i(x^i)$ are different. Third, the identity is only valid for the components defined in Minkowski's method, however the components in Landau's definition and in the unit bases often take the different form in the two systems, unless the three kinds of components reduce to equivalent under Cartesian coordinate of physical space.

TABLE I. Corresponding relations.

	Electromagnetic quantities					
Physical space :	$D_{(\mathbf{P})}^i$	$E_{(\mathbf{P})i}$	$B_{(\mathbf{P})}^i$	$H_{(\mathbf{P})i}$	$\varepsilon_{(\mathbf{P})}^{ij}$	$\mu_{(\mathbf{P})}^{ij}$
\mathbf{S} system :	D^i	E_i	B^i	H_i	ε^{ij}	μ^{ij}

Now our discussion is restricted on the condition that \mathbf{S} system is an orthogonal coordinate system with $\gamma_{ij} = h_i^2 \delta_{ij}$. Based on the corresponding relations, Eqs. (3), and (5), we

derive the permittivity and permeability in physical space:

$$\varepsilon_{(\mathbf{P})\langle ij \rangle} / \varepsilon_0 = \mu_{(\mathbf{P})\langle ij \rangle} / \mu_0 = \sum_{i'} \frac{h'_1 h'_2 h'_3}{h_{(\mathbf{P})1} h_{(\mathbf{P})2} h_{(\mathbf{P})3}} \frac{h_{(\mathbf{P})i} h_{(\mathbf{P})j}}{(h'_{i'})^2} \left| \det (\Lambda_{i'}^i)^{-1} \right| \Lambda_{i'}^i \Lambda_{i'}^j. \quad (6)$$

Here we take the transformation matrix as a function of x^i and $x'^{i'}$, *i.e.* $\Lambda_{i'}^i = \Lambda_{i'}^i(x^i, x'^{i'})$, meanwhile the independent variables should be $x_{(\mathbf{P})}^i$ and $x'^{i'}$ in the above formula.

For spherical invisibility cloak, if we select \mathbf{S}' system as spherical coordinate system, define the inverse transformation from \mathbf{S}' to \mathbf{S} system as $r' = f(r)$, $\theta' = \theta$, $\phi' = \phi$, and insert it into Eq. (6), the constitutive parameters become [15]

$$(\varepsilon_{(\mathbf{P})\langle ij \rangle} / \varepsilon_0) = (\mu_{(\mathbf{P})\langle ij \rangle} / \mu_0) = \text{diag} \left(\frac{f^2(r_{(\mathbf{P})})}{r_{(\mathbf{P})}^2 f'(r_{(\mathbf{P})})}, f'(r_{(\mathbf{P})}), f'(r_{(\mathbf{P})}) \right). \quad (7)$$

For arbitrary transformation function, it can be completely invisible, as long as $f(a) = 0$ and $f(b) = b$, where a , b are the inner and outer radii of the cloak layer, even if $f(r)$ is continuous but non-differentiable [15]. Similarly, the constitutive parameters for cylindrical cloak obtained by the mere radial transformation $r' = f(r)$, $\theta' = \theta$, $z' = z$ are [16]

$$(\varepsilon_{(\mathbf{P})\langle ij \rangle} / \varepsilon_0) = (\mu_{(\mathbf{P})\langle ij \rangle} / \mu_0) = \text{diag} \left(\frac{f(r_{(\mathbf{P})})}{r_{(\mathbf{P})} f'(r_{(\mathbf{P})})}, \frac{r_{(\mathbf{P})} f'(r_{(\mathbf{P})})}{f(r_{(\mathbf{P})})}, \frac{f(r_{(\mathbf{P})}) f'(r_{(\mathbf{P})})}{r_{(\mathbf{P})}} \right), \quad (8)$$

where the components are described by cylindrical coordinates. The arbitrary $f(r)$ will achieve completely invisible only if it satisfies the boundary condition $f(a) = 0$ and $f(b) = b$ (see the proof in Appendix B).

C. Boundary conditions for perfect invisibility cloak

The congruity of Maxwell's equations with the corresponding relations between \mathbf{S} system and \mathbf{P} space is regarded as the foundation of transformation optics. However, despite of the existence on the same form of Maxwell's equations, they won't give the same solution unless the boundary conditions of the two systems are also identical.

For example, the realization of spherical and cylindrical invisibility cloak requires the inverse transformation function satisfies $f(a) = 0$ and $f(b) = b$. The $f(a) = 0$ causes the domain $r' < a$ in \mathbf{S}' system to shrink into its original point in \mathbf{S} , so that the coordinates with radial component $r < a$ would never appear in the expression of electromagnetic fields in \mathbf{S} system. It leads to decoupling of fields between the hidden region $r_{(\mathbf{P})} < a$ and cloak layer in the \mathbf{P} space. Pendry *et al.* pointed out that the condition $f(b) = b$ makes the constitutive parameters satisfy perfectly matched layer (PML) conditions $\varepsilon_{(\mathbf{P})\langle \theta\theta \rangle} = \varepsilon_{(\mathbf{P})\langle \phi\phi \rangle} = 1/\varepsilon_{(\mathbf{P})\langle rr \rangle}$ and $\mu_{(\mathbf{P})\langle \theta\theta \rangle} = \mu_{(\mathbf{P})\langle \phi\phi \rangle} = 1/\mu_{(\mathbf{P})\langle rr \rangle}$ at the outer surface of the cloak ($r_{(\mathbf{P})} = b$) [1],

therefore the cloak is reflectiveness. However there is a more profound reason produced by the corresponding relation of the boundary conditions between \mathbf{S}' system and \mathbf{P} space.

First, we consider a spherical cloak designed by general transformation $x^i(r', \theta', \phi')$ from \mathbf{S}' to \mathbf{S} system. In \mathbf{P} space, the boundary conditions at the interface $r = b$ are

$$(\vec{E}_{(P)}|_{r=b^+} - \vec{E}_{(P)}|_{r=b^-}) \times \hat{e}_r = 0, \quad (\vec{D}_{(P)}|_{r=b^+} - \vec{D}_{(P)}|_{r=b^-}) \cdot \hat{e}_r = 0, \quad (9)$$

where \hat{e}_r acts as the normal vector of interface. The boundary conditions indicate the tangential component of \vec{E} and normal component of \vec{D} are continuous across the interface, and the two boundary conditions can completely determine $\vec{E}_{(P)}|_{r=b^-}$, if $\vec{E}_{(P)}|_{r=b^+}$ is given. Since the metric of \mathbf{P} space is continuous, the equality of components is always tenable no matter what definitions. Thus, we have the continuous relations in Minkowski's form

$$E_{(P)\theta}|_{r=b^+} = E_{(P)\theta}|_{r=b^-}, \quad E_{(P)\phi}|_{r=b^+} = E_{(P)\phi}|_{r=b^-}, \quad D_{(P)}^r|_{r=b^+} = D_{(P)}^r|_{r=b^-}. \quad (10)$$

Non-reflection means the electric field outside the cloak only has the incident wave. According to the corresponding relations in Table I, the electric field in \mathbf{S} system should also satisfy the relations

$$E_{\theta'}|_{r'=b^+} = E_{\theta}|_{r=b^+} = E_{\theta}|_{r=b^-} = \Lambda_{\theta}^{i'} E_{i'}|_{x^{i'}(b^-, \theta, \phi)}, \quad (11a)$$

$$E_{\phi'}|_{r'=b^+} = E_{\phi}|_{r=b^+} = E_{\phi}|_{r=b^-} = \Lambda_{\phi}^{i'} E_{i'}|_{x^{i'}(b^-, \theta, \phi)}, \quad (11b)$$

$$D^{r'}|_{r'=b^+} = D^r|_{r=b^+} = D^r|_{r=b^-} = \left| \det(\Lambda_{i'}^i)^{-1} \right| \Lambda_{i'}^r D^{i'}|_{x^{i'}(b^-, \theta, \phi)}, \quad (11c)$$

where $E_{\theta'}|_{r'=b^+} = E_{\theta}|_{r=b^+}$ since \mathbf{S}' and \mathbf{S} system hold the same coordinates in the domain $r > b$. Note that the boundary conditions Eqs. (9) certainly satisfy in virtual space. However the metric is discontinuous at the interface $r = b$ in \mathbf{S} system, therefore the relation of components $E_{\theta}|_{r=b^+} = E_{\theta}|_{r=b^-}$ is not trivial, but a strong limit to the metric.

By comparing the first and last term, we note that the electric field at $r' = b$ is demanded to have a fixed relation to the field at $x^{i'}(b^-, \theta, \phi)$ in \mathbf{S}' system. Because of the locality of electromagnetic fields, a certain relation of fields between different positions can not be established. Therefore (b, θ, ϕ) and $x^{i'}(b^-, \theta, \phi)$ must be the same point:

$$r'(b^-, \theta, \phi) = b, \quad \theta'(b^-, \theta, \phi) = \theta, \quad \phi'(b^-, \theta, \phi) = \phi. \quad (12)$$

This is the requirement of non-reflection. Thus, we have

$$\left(\frac{\partial \theta'}{\partial \theta} = \frac{\partial \phi'}{\partial \phi} \right) \Big|_{r=b^-} = 1, \quad \left(\frac{\partial r'}{\partial \theta} = \frac{\partial r'}{\partial \phi} = \frac{\partial \theta'}{\partial \phi} = \frac{\partial \phi'}{\partial \theta} \right) \Big|_{r=b^-} = 0.$$

Substituting it into Eqs. (11), the equalities are directly satisfied. A similar study to the boundary conditions of magnetic field can give the identical result.

In fact, the interface at $r = b$ in \mathbf{S} system generally doesn't correspond to the interface $r' = b$ in \mathbf{S}' system for an arbitrary transformation. Thus the metric is usually discontinuous across the interface, and we even hardly measure the element of length on the interface. For example, if the inverse transformation is simply $r' = f(r)$, $\theta' = \theta$, $\phi' = \phi$ as $r < b$, the elements of length on the inner and outer surface of the interface are respectively

$$dl^2|_{r=b^-} = f(b)^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad dl^2|_{r=b^+} = b^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

Nothing but $f(b) = b$ gives $dl^2|_{r=b^-} = dl^2|_{r=b^+}$ with arbitrary $d\theta$ and $d\phi$. It is also available for cylindrical cloak by a similar analysis that non-reflection requires invariance of coordinates on the interface before and after transformation.

III. ANALYTICAL EXPRESSIONS OF LIGHT-RAY AND WAVE-NORMAL RAY

A. Spherical cloak

Considering a spherical cloak, the cloak hides a homogenous-media sphere, and its center is at the origin of a spherical coordinate system. A plane wave $\vec{E}_{(P)}^{\text{in}} = E_0 e^{ik_0 r \cos \theta} \hat{e}_x$ with x polarization is incident to the cloak. It has been proved that both the fields $\vec{E}_{(P)}^{\text{int}}$ inside the internal area ($r < a$), and the scattering fields $\vec{E}_{(P)}^{\text{sc}}$ outside cloak ($r > b$) vanish if the constitutive parameters of cloak satisfy the formula Eq. (7) and the boundary conditions $f(a) = 0$, $f(b) = b$. Meanwhile the electromagnetic fields in cloak layer ($a < r < b$) have been given in [15]

$$\vec{E}_{(P)}^c = E_0 \left(f'(r) \sin \theta \cos \phi \hat{e}_r + \frac{f(r)}{r} \cos \theta \cos \phi \hat{e}_\theta - \frac{f(r)}{r} \sin \phi \hat{e}_\phi \right) e^{ik_0 f(r) \cos \theta}, \quad (13a)$$

$$\vec{D}_{(P)}^c = \varepsilon_0 E_0 \frac{f(r)}{r} \left(\frac{f(r)}{r} \sin \theta \cos \phi \hat{e}_r + f'(r) \cos \theta \cos \phi \hat{e}_\theta - f'(r) \sin \phi \hat{e}_\phi \right) e^{ik_0 f(r) \cos \theta}, \quad (13b)$$

$$\vec{H}_{(P)}^c = \frac{E_0}{\eta_0} \left(f'(r) \sin \theta \sin \phi \hat{e}_r + \frac{f(r)}{r} \cos \theta \sin \phi \hat{e}_\theta + \frac{f(r)}{r} \cos \phi \hat{e}_\phi \right) e^{ik_0 f(r) \cos \theta}, \quad (13c)$$

$$\vec{B}_{(P)}^c = \frac{E_0 f(r)}{cr} \left(\frac{f(r)}{r} \sin \theta \sin \phi \hat{e}_r + f'(r) \cos \theta \sin \phi \hat{e}_\theta + f'(r) \cos \phi \hat{e}_\phi \right) e^{ik_0 f(r) \cos \theta}, \quad (13d)$$

where $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ and $k_0 = \omega/c$ are the impedance and wave number of vacuum respectively. Since only fields in \mathbf{P} space are discussed in this section, we omit the subscript “(P)” in all the expressions of coordinate, but still retain it in the expressions of electromagnetic quantities. According to above fields, we obtain the time-averaged Poynting vector

$$\langle \vec{S}_{(P)}^c \rangle = \frac{1}{2} \text{Re}(\vec{E}_{(P)}^c \times \vec{H}_{(P)}^c) = \frac{E_0^2 f(r)}{2\eta_0 r} \left(\frac{f(r)}{r} \cos \theta \hat{e}_r - f'(r) \sin \theta \hat{e}_\theta \right), \quad (14)$$

It can be verified that the Poynting vector satisfies energy conservation $\nabla \cdot \langle \vec{S}_{(P)}^c \rangle = 0$ [23]. Supposing the medium of cloak is non-dispersive, the time-averaged energy density of electromagnetic fields has the simple form

$$\langle W_{(P)}^c \rangle = \frac{1}{2} \text{Re} \left(\frac{1}{2} \vec{E}_{(P)}^c \cdot \vec{D}_{(P)}^c + \frac{1}{2} \vec{H}_{(P)}^c \cdot \vec{B}_{(P)}^c \right) = \frac{\varepsilon_0 E_0^2 f(r)^2 f'(r)}{2r^2}. \quad (15)$$

Thus, the ray (energy) velocity in cloak is given as [23, 24]

$$\vec{v}_{(P)ray} = \frac{\langle \vec{S}_{(P)}^c \rangle}{\langle W_{(P)}^c \rangle} = c \left(\frac{1}{f'(r)} \cos \theta \hat{e}_r - \frac{r}{f(r)} \sin \theta \hat{e}_\theta \right). \quad (16)$$

The ray vector is [24]

$$\vec{s}_{(P)} = \frac{\vec{v}_{(P)ray}}{\omega} = k_0^{-1} \left(\frac{1}{f'(r)} \cos \theta \hat{e}_r - \frac{r}{f(r)} \sin \theta \hat{e}_\theta \right). \quad (17)$$

In terms of the eikonal $\psi = k_0 f(r) \cos \theta$, the geometric wave-front, $\psi = \text{constant}$, is

$$f(r) \cos \theta = \text{const}. \quad (18)$$

Then, we can get the wave vector $\vec{k}_{(P)}$, which is orthogonal to the wave-front, in the cloak

$$\vec{k}_{(P)} = \nabla \psi = k_0 \left(f'(r) \cos \theta \hat{e}_r - \frac{f(r)}{r} \sin \theta \hat{e}_\theta \right). \quad (19)$$

In the anisotropic media, the ray vector (time-averaged Poynting vector) is tangent to light-ray, however the wave vector is tangent to wave-normal ray, where \vec{s} and \vec{k} are generally not parallel to each other. The parametric equations of light-ray can be written as

$$\frac{dr}{d\lambda} = s_{(P)}^r = \frac{1}{k_0 f'(r)} \cos \theta, \quad (20a)$$

$$\frac{d\theta}{d\lambda} = s_{(P)}^\theta = -\frac{1}{k_0 f(r)} \sin \theta, \quad (20b)$$

$$\frac{d\phi}{d\lambda} = s_{(P)}^\phi = 0, \quad (20c)$$

where λ is the parameter of light-ray parametric equation, and the components of ray vector employed above are contravariant in holonomic spherical coordinate bases. The ϕ maintains its initial value ϕ_0 through out a light-ray in terms of Eq. (20c). By eliminating λ from Eq. (20a) and Eq. (20b), we have

$$\int \frac{f'(r)}{f(r)} dr = - \int \frac{\cos \theta}{\sin \theta} d\theta.$$

Then the analytical expression of light-ray is obtained

$$f(r) \sin \theta = \text{const}, \quad \phi = \phi_0. \quad (21)$$

Here, the integral constant and ϕ_0 depend on the incident point on the outer surface of the cloak.

A similar analysis is used to get the parametric equations for the wave-normal ray

$$\frac{dr}{d\lambda} = k_{(P)}^r = k_0 f'(r) \cos \theta, \quad (22a)$$

$$\frac{d\theta}{d\lambda} = k_{(P)}^\theta = -k_0 \frac{f(r)}{r^2} \sin \theta, \quad (22b)$$

$$\frac{d\phi}{d\lambda} = k_{(P)}^\phi = 0. \quad (22c)$$

Thus, the wave-normal ray equation is

$$\exp \left[\int \frac{f(r)}{r^2 f'(r)} dr \right] \sin \theta = \text{const}, \quad \phi = \phi_0. \quad (23)$$

According to Eq. (16), when r closes to a , $f(r)$ tends to 0, and the modulus of ray velocity increases beyond vacuum light speed c and tends to infinity which is against the law of causality. The contradiction indicates that the assumption of non-dispersive cloak is unpractical. In the presence of dispersion, the time-averaged energy density has a more complicated formula. It includes the derivative of constitutive parameters with respect to ω [24], and then the superluminal result should disappear. More detailed discussion to this point can be found in [18, 19]. Nevertheless, the expressions in Eqs. (16, 17) still exactly represent the direction of the “real ray velocity”, therefore we will retain the expressions of $\vec{v}_{(P)ray}$ and $\vec{s}_{(P)}$, but limit them only to stand for the tangent vector of light-ray.

B. Cylindrical cloak

Considering a cylindrical cloak with indefinite length, its axis is z axis of a cylindrical coordinate system. The incident wave is a transverse-electric(TE) wave $\vec{E}_{(P)}^{\text{in}} = E_0 e^{ik_0 r \cos \theta} \hat{e}_z$ with z polarization propagating towards x direction. If the cloak hold the perfect invisible parameters expressed in Eq. (8), the calculation in Appendix B gives $\vec{E}_{(P)}^{\text{int}} = 0$ in the inner hidden area ($r < a$), and the scattering $\vec{E}_{(P)}^{\text{sc}} = 0$ outside the cloak ($r > b$). Meanwhile the fields in the cloak layer are

$$\vec{E}_{(P)}^c = E_0 e^{ik_0 f(r) \cos \theta} \hat{e}_z, \quad (24a)$$

$$\vec{D}_{(P)}^c = \varepsilon_0 E_0 \frac{f'(r)f(r)}{r} e^{ik_0 f(r) \cos \theta} \hat{e}_z, \quad (24b)$$

$$\vec{H}_{(P)}^c = -\frac{E_0}{\eta_0} \left(f'(r) \sin \theta \hat{e}_r + \frac{f(r)}{r} \cos \theta \hat{e}_\theta \right) e^{ik_0 f(r) \cos \theta}, \quad (24c)$$

$$\vec{B}_{(P)}^c = -\frac{E_0}{c} \left(\frac{f(r)}{r} \sin \theta \hat{e}_r + f'(r) \cos \theta \hat{e}_\theta \right) e^{ik_0 f(r) \cos \theta}. \quad (24d)$$

Thus, the time-averaged Poynting vector is

$$\langle \vec{S}_{(P)}^c \rangle = \frac{E_0^2}{2\eta_0} \left(\frac{f(r)}{r} \cos \theta \hat{e}_r - f'(r) \sin \theta \hat{e}_\theta \right). \quad (25)$$

and the time-averaged energy density of electromagnetic fields under the assumption of non-dispersion is

$$\langle W_{(P)}^c \rangle = \frac{\varepsilon_0 E_0^2 f(r) f'(r)}{2r}. \quad (26)$$

By calculation, the ray vector and wave vector in the cylindrical cloak have the identical form in the spherical cloak as shown in Eq. (17) and Eq. (19). Whereas, the variables r and θ in the spherical cloak are spherical coordinates, while in cylindrical cloak they represent cylindrical coordinates.

Thus, following the same calculation as in spherical cloak, we derive the light-ray expression in cylindrical cloak

$$f(r) \sin \theta = \text{const}, \quad z = z_0, \quad (27)$$

and the wave-normal ray equation

$$\exp \left[\int \frac{f(r)}{r^2 f'(r)} dr \right] \sin \theta = \text{const}, \quad z = z_0. \quad (28)$$

C. Examples of four types of cloak

In this section, we apply the obtained expressions of light-ray and wave-normal ray to four cloaks constructed by different transformation functions. Type 1 is the linear transformation function $f_1(r) = [b/(b-a)](r-a)$ provided in [1, 4]. Type 2 is $f_2(r) = b[(r-a)/(b-a)]^2$, and type 3 is $f_3(r) = b - b[(b-r)/(b-a)]^2$. Both type 2 and 3 are from [15]. Type 4 is $f_4(r) = (b/2a)[2a - b + \sqrt{b^2 - 4ab + 4ar}]$ provided in [16]. They all meet the invisibility condition $f(a) = 0$, $f(b) = b$ (see Fig. 2).

We construct four spherical cloaks with the four transformations. Each row in Fig. 3 displays a type of cloak corresponding to $f_1(r)$ to $f_4(r)$ from top to bottom. The first column displays the components of electric fields $E_{(P)(x)}$ in the $x = 0$ plane. The second column displays the light-rays and wave-fronts of each cloak. The third column shows the wave-normal rays and wave-fronts which are always perpendicular to each other.

From Fig. 3, some characteristics of each cloak can be concluded. In Fig. 3(b), the light-rays and wave-fronts hold equal spacing respectively, which is due to its linear transformation $f_1(r)$. For the second cloak, because $f_2'(r) \rightarrow 0$ as $r \rightarrow a$, the θ component of energy flow decreases in $r \rightarrow a$ so fast that the energy traveling in cloak mainly concentrates at the outer boundary. Thereby the light-rays mostly compress near the outer boundary shown in Fig.

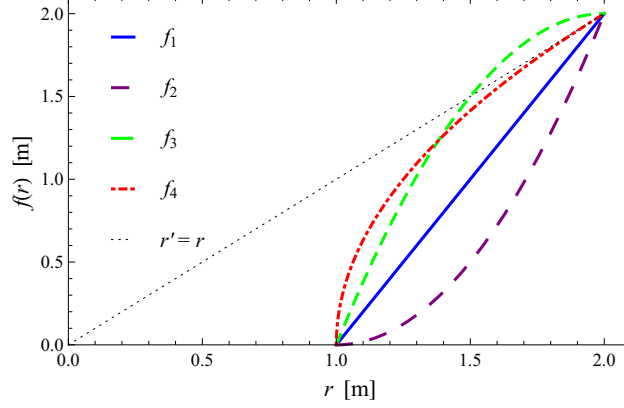


FIG. 2. Transformation functions. The inner and outer radii of cloak are $a = 1\text{m}$ and $b = 2\text{m}$ respectively.

3(e). For the third cloak, since $f'_3(b) = 0$, the light-rays are condensed to inner boundary as shown in Fig. 3(h). In addition, $r = b$ as $k_{(P)r} \equiv 0$, that is to say, the circle $r = b$ becomes a closed wave-normal ray, and the wave-normal rays inside cloak are tangent to the outer interface $r = b$, as shown in Fig. 3(i). For the fourth cloak, its ε and μ are continuous to vacuum across outer interface due to $f'_4(b) = 1$. Therefore, all the light-rays, wave-normal rays and wave-fronts are smooth when they pass through the outer boundary of cloak as shown in Fig. 4(k)(l).

D. Verifying the light-ray equation is the solution of geodesic equations

In this section, We will verify that the light-ray analytical expression (21) in a spherical cloak satisfies the geodesic equation in \mathbf{S} system of virtual space, which is [17]

$$\frac{dk^i}{d\lambda} + \Gamma_{jl}^i k^j k^l = 0, \quad (29)$$

where $\Gamma_{jl}^i = \frac{1}{2}\gamma^{im}(\gamma_{mj,l} + \gamma_{ml,j} - \gamma_{jl,m})$ is the Christoffel symbol in 3-D space. In \mathbf{S} system, the spatial metric is

$$\gamma_{ij} = \text{diag}(f'(r)^2, f^2(r), f^2(r) \sin^2 \theta). \quad (30)$$

Substituting it into Christoffel symbol, we have

$$\left\{ \begin{array}{l} \Gamma_{rr}^r = \frac{f''(r)}{f'(r)}, \quad \Gamma_{\theta\theta}^r = -\frac{f(r)}{f'(r)}, \quad \Gamma_{\phi\phi}^r = -\frac{f(r)}{f'(r)} \sin^2 \theta, \\ \Gamma_{\theta r}^\theta = \frac{f'(r)}{f(r)}, \quad \Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta, \\ \Gamma_{\phi r}^\phi = \frac{f'(r)}{f(r)}, \quad \Gamma_{\phi\theta}^\phi = \cot \theta, \end{array} \right. \quad (31)$$

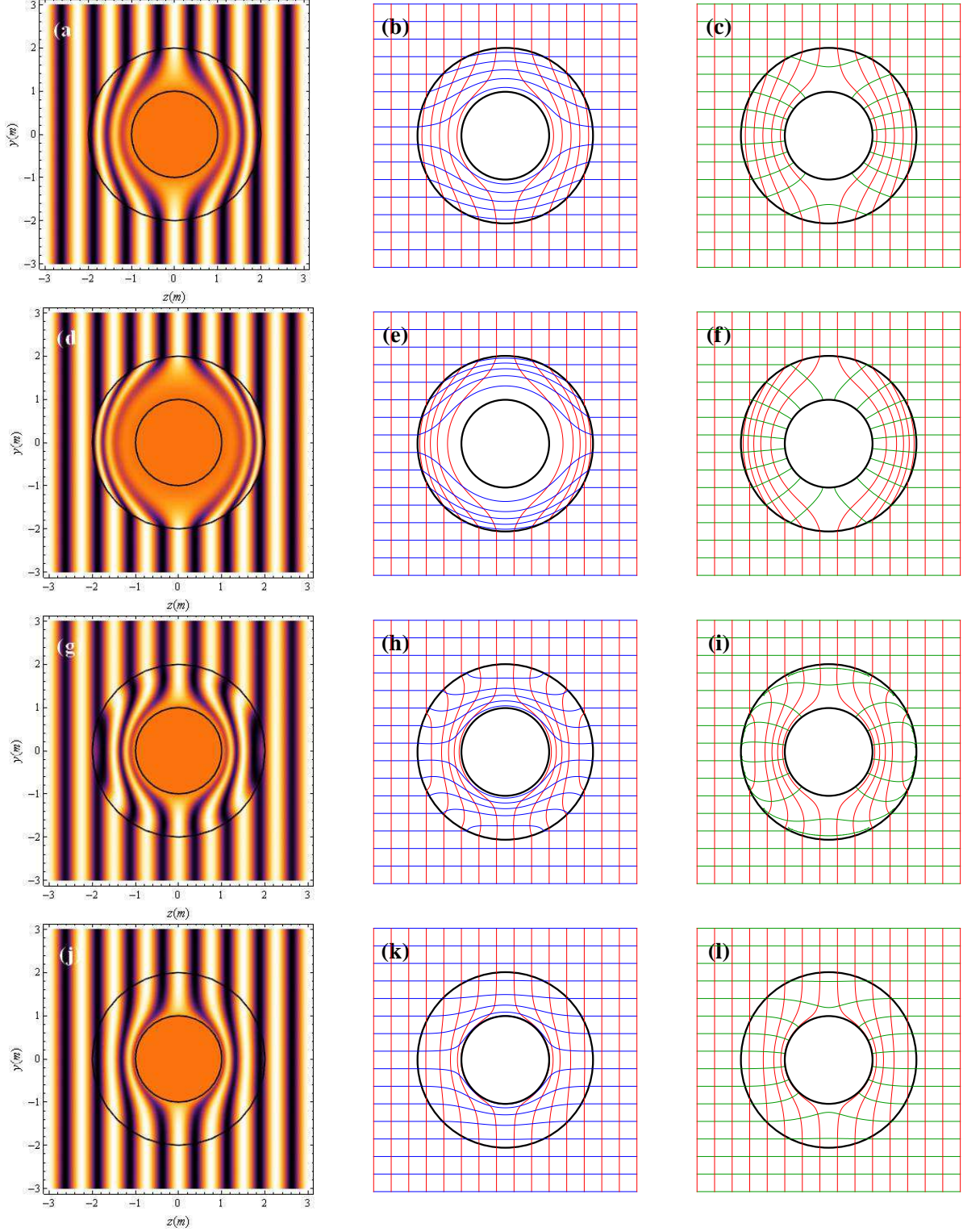


FIG. 3. Lights travel in $x = 0$ plane of four space where four types of cloak are placed respectively. Field distributions of $E_{(P)\langle x \rangle}$ of each cloak are shown in (a)(d)(g)(j). In (b)(e)(h)(k), blue lines denote light-rays, red lines denote wave-fronts. In (c)(f)(i)(l), green lines denote wave-normal rays, red lines still denote wave-fronts.

and other components are all null. Then, the geodesic equations become

$$\frac{dk^r}{d\lambda} + \frac{f''(r)}{f'(r)} k^r k^r - \frac{f(r)}{f'(r)} k^\theta k^\theta - \frac{f(r)}{f'(r)} \sin \theta k^\phi k^\phi = 0, \quad (32a)$$

$$\frac{dk^\theta}{d\lambda} + 2 \frac{f'(r)}{f(r)} k^\theta k^r - \sin \theta \cos \theta k^\phi k^\phi = 0, \quad (32b)$$

$$\frac{dk^\phi}{d\lambda} + 2 \frac{f'(r)}{f(r)} k^\phi k^r + \cot \theta k^\theta k^\phi = 0. \quad (32c)$$

Tracing the ray path shown in Eq. (21), the differentials of coordinates satisfy

$$d\theta = -\frac{f'(r)}{f(r)} \tan \theta dr, \quad d\phi = 0.$$

Substituting them into the relation between parameter λ and element of length $dl = (\gamma_{ij} dx^i dx^j)^{1/2}$, which is $d\lambda = -(c\sqrt{-g_{00}}/\omega)dl$ [25], we have

$$d\lambda = \frac{cf'(r)}{\omega \cos \theta} dr.$$

Therefore, the wave vector can be expressed as

$$k^r = \frac{dr}{d\lambda} = k_0 \frac{\cos \theta}{f'(r)}, \quad k^\theta = \frac{d\theta}{d\lambda} = -k_0 \frac{\sin \theta}{f(r)}, \quad k^\phi = 0. \quad (33)$$

This is the wave vector in **S** system, its relation with the wave vector (19) in **P** space will be discussed in section 4.2. Substituting k^i into Eqs. (32), then the left sides of the equations reduce to zero, equal to the right side. As a result, we find out that Eq. (21) is a particular analytical solution of the geodesic equations.

E. Physical interpretation of light-ray equation

We have verified the light-ray (21) denotes a geodesic expressed by curved coordinates **S** of virtual space, thus the physical meaning of the light-ray equation is quite clear.

Since the virtual space is a flat vacuous space with a zero curvature, the light-ray trajectory which is also the geodesic in virtual space is just a straight line. Fig. 4(a) shows the light-ray with a Cartesian coordinate system (**S'** system). Through the inverse transformation $r' = f(r)$, where $f(a) = 0$, $f(b) = b$ inside the domain $r < b$ in **S'** system, we obtain a new system, **S** system. Through the transformation, the domain $r' < a$ in **S'** system shrinks into origin, that is, the coordinates with component $r < a$ do not exist in **S** system. Fig. 4(b) shows the result in **S** system. The coordinate lines, whose tangent vectors are bases, are defined by

$$\text{Latitude lines : } y = r \sin \theta = f^{-1}(r') \sin \theta' = \text{const.} \quad (34a)$$

$$\text{Longitude lines : } x = r \cos \theta = f^{-1}(r') \cos \theta' = \text{const.} \quad (34b)$$

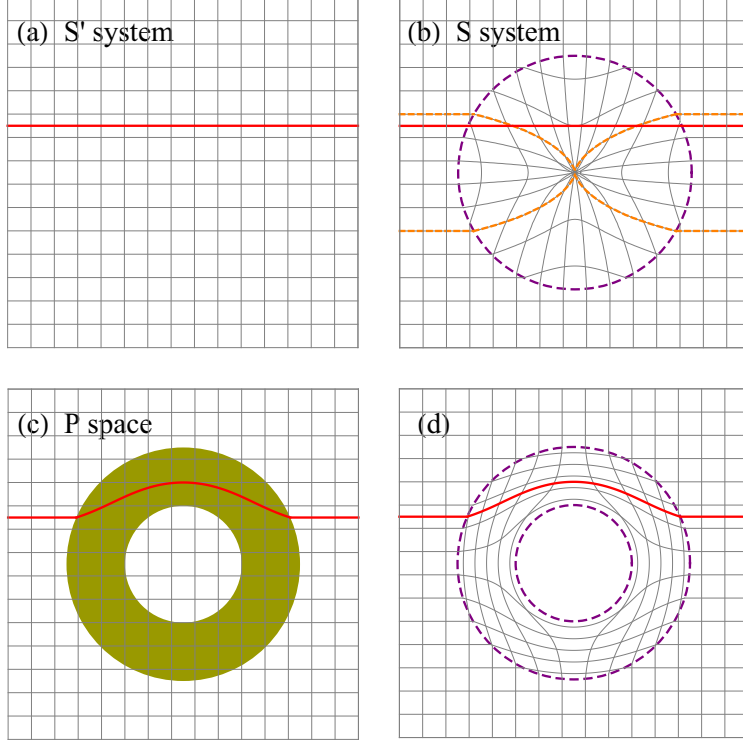


FIG. 4. Interpretation of light-rays. (a) The red line denotes a light-ray propagating in virtual space. The meshes are the coordinate lines of \mathbf{S}' system. (b) The red line is still the light-ray in virtual space. The curved meshes are the coordinate lines of \mathbf{S} system. Two orange dashed lines are the boundaries inside which latitude lines shrink into the origin point. (c) Light-ray travels in \mathbf{P} space with Cartesian coordinate meshes. (d) The mapping of lines with the expression of geodesic in \mathbf{S} system to \mathbf{P} space.

where $f^{-1}(r')$ is the inverse function of $f(r)$. We observe that the coordinate lines cave towards the origin, especially the lines whose $|x| < a$ or $|y| < a$ outside the interface $r = b$ cave directly into the origin when they pass through the interface. Therefore, the origin becomes an ambiguous point of coordinates in \mathbf{S} system. However, the coordinate transformation doesn't change the curvature of virtual space, and the light trajectories, *i.e.* geodesics, are still straight with the expression in \mathbf{S} system

$$y' = r' \sin \theta' = f(r) \sin \theta = \text{const.} \quad (35)$$

It corresponds to the light-ray equation that we have obtained previously.

The light-ray function maps to \mathbf{P} space with the same form $f(r_{(\mathbf{P})}) \sin \theta_{(\mathbf{P})} = \text{const.}$ Since \mathbf{P} space is also a flat space, the light-ray becomes convex as shown in Fig. 4(c). Meanwhile, the constitutive parameters both in spherical and cylindrical cloak have singular components

with zero or infinite value at inner surface for $f(a) = 0$. In other words, the geometrical singularity of coordinates turns into the singularity of material, when mapping from \mathbf{S} system to \mathbf{P} space.

We use an analogy to describe the process. The virtual space can be compared to a piece of flat elastic cloth with straight meshes on it. The meshes look like \mathbf{S}' system. If we fix the boundary $r = b$, and drag the whole region $r < a$ into the center point, the meshes on the cloth are curved as the coordinate lines of \mathbf{S} system. Then, we draw a straight line on the cloth just as the red line in Fig 4(b) which plays the role of light-ray in \mathbf{S} . After that, we loose the piece of cloth held into the center, and let it return flat. Then the meshes also returns to straightness, but the red line we have drawn becomes convex exactly as the ray propagating in \mathbf{P} space.

Now we should clarify the meaning of Fig. 4(d). Sometimes the meshes in this figure are interpreted as the coordinate of \mathbf{S} system [4]. This interpretation gives rise to an illusion that \mathbf{S} system is a real curved space, as well as the light trajectories. However, the meshes in Fig. 4(d) are neither coordinate lines nor geodesics no matter in \mathbf{S} system or in \mathbf{P} space. The only reasonable interpretation is that they are the curves in \mathbf{P} space directly mapping from the straight lines in \mathbf{S} system, and can describe the light trajectories in \mathbf{P} space [7, 9].

IV. DUALITY PRINCIPLE IN ANISOTROPIC MEDIA

In virtual space, light propagates in straight lines, and the directions of wave vectors and ray vectors are always identical, however when mapping to physical space, they are not parallel to each other any more. So what exactly causes the split? Why does the light-ray instead of not the wave-normal ray in \mathbf{P} space inherit its expression in \mathbf{S} system? In this section, we will try to answer these questions by using duality principle.

In charge free anisotropic media, electromagnetic fields have the following duality rule [23]

$$\text{Duality rule} \left\{ \begin{array}{cccccccc} \vec{D} & \vec{E} & \vec{B} & \vec{H} & \vec{\epsilon} & \vec{\mu} & \vec{k}/\omega & v_p \\ \vec{E} & \vec{D} & \vec{H} & \vec{B} & \vec{\epsilon}^{-1} & \vec{\mu}^{-1} & \vec{s}\omega & 1/v_{ray} \end{array} \right.$$

where $v_p = \omega/k$ is the phase speed. In terms of the above systematic interchange relations, the either side satisfies a set of formulas for plane waves [24]:

$$\begin{aligned}
\vec{D} &= \vec{\varepsilon} \cdot \vec{E} = -\frac{1}{\omega} \vec{k} \times \vec{H} & \vec{E} &= \vec{\varepsilon}^{-1} \cdot \vec{D} = -\omega \vec{s} \times \vec{B} \\
\vec{B} &= \vec{\mu} \cdot \vec{H} = \frac{1}{\omega} \vec{k} \times \vec{E} & \Leftrightarrow \text{duality} \Rightarrow & \vec{H} = \vec{\mu}^{-1} \cdot \vec{B} = \omega \vec{s} \times \vec{D} \\
\vec{k}/\omega &= \hat{k}/v_p = \frac{\vec{D} \times \vec{B}}{\vec{E} \cdot \vec{D}} & \vec{s}\omega &= v_{ray} \hat{s} = \frac{\vec{E} \times \vec{H}}{\vec{D} \cdot \vec{E}}
\end{aligned} \tag{36}$$

According to the left side of the formulas, the wave equation is founded as

$$\vec{k} \times \left[\vec{\mu}^{-1} \cdot (\vec{k} \times \vec{E}) \right] + \omega^2 \vec{\varepsilon} \cdot \vec{E} = 0. \tag{37}$$

Concerning the special case of impedance matched material $\vec{\varepsilon}/\varepsilon_0 = \vec{\mu}/\mu_0 = \vec{n}$, the dispersion relation, which are also called wave-vector eikonal equation, is (derived in detail in Appendix C)[4]

$$\bar{n}^{ij} k_i k_j - \gamma \left(\frac{\omega}{c} \right)^2 \det(\bar{n}^{ij}) = \bar{n}_{ij} k^i k^j - \frac{1}{\gamma} \left(\frac{\omega}{c} \right)^2 \det(\bar{n}_{ij}) = 0. \tag{38}$$

Here, there are two ambiguities to be clarify. First, the expression given in [4] is based on Cartesian coordinates, so γ reduces to unit, while the expression here is valid in any curved coordinates. Second, \bar{n}^{ij} and \bar{n}_{ij} presented above express the components in holonomic coordinate bases (Landau's definition), however, the dispersion relations expressed either in Minkowski's definition or in anholonomic unit basis are a little different from Eq. (38) (see Appendix C). In [4], the three kinds of expression are congruent with each other in Cartesian coordinates.

Using the duality principle, we can easily obtain the dual form of wave equation

$$\vec{s} \times \left[\vec{\mu} \cdot (\vec{s} \times \vec{D}) \right] + \frac{1}{\omega^2} \vec{\varepsilon}^{-1} \cdot \vec{D} = 0. \tag{39}$$

Thus, the “ray-vector eikonal equation” is

$$(\bar{n}^{-1})^{ij} s_i s_j - \gamma \left(\frac{c}{\omega} \right)^2 \det[(\bar{n}^{-1})^{ij}] = (\bar{n}^{-1})_{ij} s^i s^j - \frac{1}{\gamma} \left(\frac{c}{\omega} \right)^2 \det[(\bar{n}^{-1})_{ij}] = 0, \tag{40}$$

where $(\bar{n}^{-1})^{ij}$ is the inverse matrix of \bar{n}_{ij} , and it is easy to verify that its corresponding covariant components $(\bar{n}^{-1})_{ij}$ are also the inverse matrix of \bar{n}^{ij} .

A. Eikonal equation in curved coordinate system **S**

We first consider the relation between k^i and s^i in curved coordinate system **S**. In **S** system, the relative constitutive parameter is $\bar{n}^{ij} = (\bar{n}^{-1})^{ij} = \gamma^{ij}$. Substituting it into Eq. (38) and Eq. (40), we get the wave-vector eikonal equation in **S** system

$$\gamma^{ij} k_i k_j - \left(\frac{\omega}{c} \right)^2 = \gamma_{ij} k^i k^j - \left(\frac{\omega}{c} \right)^2 = 0, \tag{41}$$

and the ray-vector eikonal equation

$$\gamma^{ij} s_i s_j - \left(\frac{c}{\omega}\right)^2 = \gamma_{ij} s^i s^j - \left(\frac{c}{\omega}\right)^2 = 0. \quad (42)$$

The two equations indicate that $|\vec{k}| = k_0$ and $|\vec{s}| = 1/k_0$. By considering the general relation $\vec{k} \cdot \vec{s} = \gamma_{ij} k^i s^j = 1$ [24], we can conclude that the directions of \vec{k} and \vec{s} completely coincide, and they have the relation $k_i = k_0^2 s_i$. This result is actually not marvelous, since the electromagnetic wave is merely propagating in a vacuous space, despite the space is expressed in a curvilinear coordinate system with non-unit metric γ_{ij} .

B. Eikonal equation in physical space \mathbf{P}

In physical space, the media of the cloak are inhomogeneous and waves traveling in it are no longer plane waves. Nevertheless, we can verify the fields obtained both in the spherical and cylindrical cloak still satisfy Eqs. (36, 37, 39). The relative constitutive parameter of the cloak is $\bar{n}_{(\mathbf{P})}^{ij} = \sqrt{\gamma/\gamma_{(\mathbf{P})}} \gamma^{ij}$, and its inverse is $(\bar{n}_{(\mathbf{P})}^{-1})_{ij} = \sqrt{\gamma_{(\mathbf{P})}/\gamma} \gamma_{ij}$. Substituting them into Eq. (38) and Eq. (40), we obtain the wave-vector eikonal equation in \mathbf{P} space

$$\gamma^{ij} k_{(\mathbf{P})i} k_{(\mathbf{P})j} - \left(\frac{\omega}{c}\right)^2 = (\gamma_{(\mathbf{P})ik} \gamma_{(\mathbf{P})il} \gamma^{kl}) k_{(\mathbf{P})}^i k_{(\mathbf{P})}^j - \left(\frac{\omega}{c}\right)^2 = 0, \quad (43)$$

and the ray-vector eikonal equation in \mathbf{P} space

$$\gamma_{ij} s_{(\mathbf{P})}^i s_{(\mathbf{P})}^j - \left(\frac{c}{\omega}\right)^2 = (\gamma_{(\mathbf{P})}^{ik} \gamma_{(\mathbf{P})}^{jl} \gamma_{kl}) s_{(\mathbf{P})i} s_{(\mathbf{P})j} - \left(\frac{c}{\omega}\right)^2 = 0. \quad (44)$$

Comparison of Eqs. (43, 44) and Eqs. (41, 42) manifests that the wave-vector eikonal equations expressed by covariant components k_i have the same form in \mathbf{P} space and \mathbf{S} system. However if we write the wave-vector eikonal equations with contravariant components k^i , the expressions in \mathbf{P} and in \mathbf{S} are different. Similarly, the expressions with contravariant components s^i have the same form in the two systems, yet the expressions with covariant components s_i are different. Considering the relation $\vec{k}_{(\mathbf{P})} \cdot \vec{s}_{(\mathbf{P})} = 1$ simultaneously, we can get

$$k_{(\mathbf{P})i} = k_0^2 \gamma_{ij} s_{(\mathbf{P})}^j, \quad s_{(\mathbf{P})}^i = k_0^{-2} \gamma^{ij} k_{(\mathbf{P})j}. \quad (45)$$

This relation is easily validated according to Eq. (17) and (19). It is also valid in \mathbf{S} system which means \vec{k} is parallel to \vec{s} , however this does not make sense in invisibility cloak.

These results also can be verified with canonical equations given by [4]. If we define Hamiltonian by wave-vector eikonal:

$$H = \gamma^{ij} k_{(\mathbf{P})i} k_{(\mathbf{P})j} - \left(\frac{\omega}{c}\right)^2, \quad (46)$$

and according to the first equation of canonical equations

$$\frac{dx_{(P)}^i}{d\tau} = \frac{\partial H}{\partial k_{(P)i}}, \quad \frac{dk_{(P)i}}{d\tau} = -\frac{\partial H}{\partial x_{(P)}^i}, \quad (47)$$

where τ is the parameter of canonical equations, we obtain the tangent vector of light-ray $dx_{(P)}^i/d\tau = 2\gamma^{ij}k_{(P)j}$. It reveals the tangent of light-ray is exactly towards the direction of $s_{(P)}^i$ given in Eq. (45). If we define another Hamiltonian by ray-vector eikonal:

$$\tilde{H} = (\gamma_{(P)}^{ik}\gamma_{(P)}^{jl}\gamma_{kl})s_{(P)i}s_{(P)j} - \left(\frac{c}{\omega}\right)^2, \quad (48)$$

and modify the canonical equations into

$$\frac{dx_{(P)}^i}{d\tau} = \frac{\partial \tilde{H}}{\partial s_{(P)i}}, \quad \frac{ds_{(P)i}}{d\tau} = -\frac{\partial \tilde{H}}{\partial x_{(P)}^i}, \quad (49)$$

we would have

$$\frac{dx_{(P)}^i}{d\tau} = 2\gamma_{(P)}^{ik}\gamma_{(P)}^{jl}\gamma_{kl}s_{(P)j} = 2k_0^{-2}k_{(P)}^i. \quad (50)$$

As a result, the modified canonical equations depict the trace of wave-normal rays.

Actually, if we express k_i and s^i with D^i , B^i , E_i , H_i in Minkowski's definition, we have

$$k_i = \omega \frac{e_{ijk}D^jB^k}{E_iD^i}, \quad s^i = \frac{1}{\omega} \frac{e^{ijk}D_jB_k}{E_iD^i}. \quad (51)$$

The expressions apply to both **S** system and **P** space. According to the relations in Table I, we can get two new corresponding relations between **S** and **P**:

$$k_i \Leftrightarrow k_{(P)i}, \quad s^i \Leftrightarrow s_{(P)}^i.$$

We note that k^i and $k_{(P)}^i$ do not correspond to each other, and neither do s_i and $s_{(P)i}$. In other words, covariant components of wave vector k_i in **S** system map to wave vector $k_{(P)i}$ in **P** space, yet contravariant components of wave vector k^i ($= k_0^2 s^i$) map to ray vector $s_{(P)}^i$ in **P** space. Although k_i , s^i denote a same orientation in **S** system, $k_{(P)i}$, and $s_{(P)}^i$ denote two different orientations in **P** space. In addition, the contravariant components $s^i = dx^i/d\lambda$ directly represent the directional change of coordinates dx^i , therefore light-ray equation in **P** space inherits itself's expression in **S**. By contrast, k_i can not directly evince the change of dx^i , so the expression of wave-normal ray takes a new form in **P** space.

According to the dispersion relation (43), we can obtain the group velocity in the cloak through applying the formula $\vec{v}_g = \nabla_{\vec{k}}\omega$, where $\nabla_{\vec{k}}$ is the divergence operator in \vec{k} -space. Supposing the medium is non-dispersive, the group velocity takes the identical expression of ray velocity (16). It means the group velocity and ray velocity are congruent in the cloak.

As commented in section 3.1, the contradiction of super velocity of light takes into account the existence of dispersion. However, the dispersive processes of ε and μ are different, so

the case of impedance matching established in a particular frequency would hardly retain when frequency changes. Therefore, the dispersion relation of impedance matched material (38) is unavailable when we use $\vec{v}_g = \nabla_{\vec{k}}\omega$ to calculate the group velocity in dispersive case, because Eq. (38) can not display the differences between the derivative of ε and μ with respect to ω .

V. DESIGN OF THE INVISIBILITY CLOAK WITH RAYS SATISFYING HARMONIC FUNCTION

In Ref. [2], Leonhardt applied Жуковский function $\zeta(z) = z + 1/z$, where $z = r e^{i\theta}$, to a conformal mapping, and obtained an inhomogeneous but isotropic medium which can realize an invisibility device in geometric limit. In the media, the families of light-rays and of wave-fronts are the imaginary and real part of $\zeta(z)$ respectively. In addition, they are both harmonic functions and orthogonal to each other.

Using the expression of light-ray (27), we can design an invisibility cloak by means of transformation-optics method, which gives the same light trajectories as in Leonhardt's device. However, we start our work with a more general target to find the general form of transformation function $f(r)$ which causes the light-rays in cylindrical cloak to satisfy harmonic function. Substitution of light-ray (27) into Laplace's equation yields

$$\nabla_{2-D} [f(r) \sin \theta] = 0. \quad (52)$$

Then, it reduces to Euler's differential equation, $r^2 f''(r) + r f'(r) - f(r) = 0$, therefore the general solution is $f(r) = C_1(r^2 - C_2)/r$. The function has a zero point $r = C_2^{1/2} > 0$, which is the most important condition to construct an invisibility cloak. If we set $C_1 = b^2/(b^2 - a^2)$ and $C_2 = a^2$, the transformation function becomes

$$f(r) = \frac{b^2}{b^2 - a^2} \frac{r^2 - a^2}{r}. \quad (53)$$

The result evinces that each solution corresponds to an invisibility cloak with a spacial pair of inner and outer radii. In the process of derivation, we didn't command $f(a) = 0, f(b) = b$, however the result indicates that the form of harmonic functions $V(r, \theta) = f(r) \sin \theta$ is unique, and it always can realize invisibility cloak. For convenience, we omit the coefficient of $f(r)$ in the following derivation which doesn't influence the shape of rays.

$V = \text{const}$ represents the family of light-rays in the cloak. Let V be the imaginary part of an analytical function $F(z)$. Using Cauchy-Riemann condition, we obtain the real part $U(r, \theta) = [(r^2 + a^2)/r] \cos \theta$. The $U = \text{const}$ stands for a family of curved surfaces, which is orthogonal to the energy flows. The analytical function $F(z)$ is

$$F(z) = z + \frac{a^2}{z}. \quad (54)$$

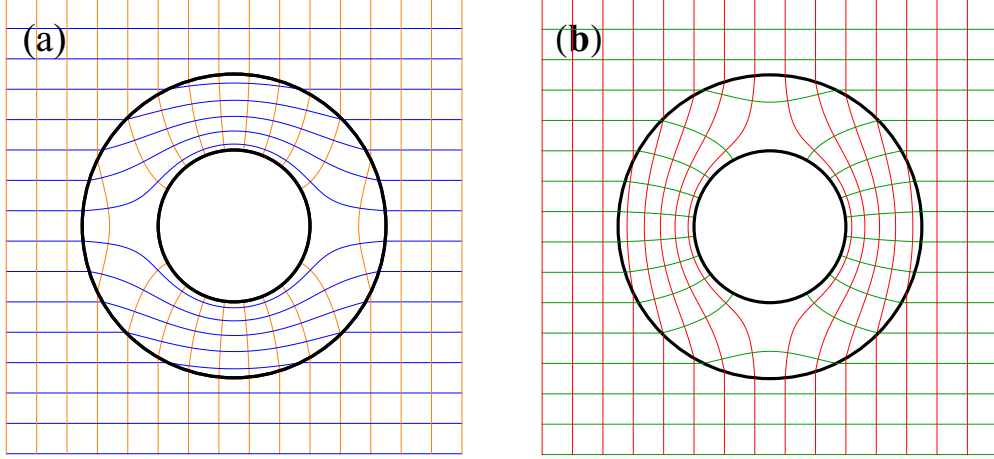


FIG. 5. Cloak with rays satisfying harmonic equation. (a) Blue lines denote light-rays, orange lines denote the surfaces orthogonal to light-rays. In cloak layer, they are painted by Eq. (54). (b) Green lines denote wave-normal rays, red lines denote wave-fronts. In cloak layer, they are painted by Eq. (55).

If we simply set $z' = z/a$, then $F(z) = a\zeta(z')$, where $\zeta(z) = z + 1/z$ is precisely Жуковский function. It is exactly the conformal mapping which Leonhardt used to design the invisibility devices [2]. The complex analytical function is a compact way to describe the ray path by the meaning of real and imaginary part explained above, as is shown in Fig. 5(a).

Regarding to the transformation (53), the eikonal $\psi = k_0 f(r) \cos \theta$ is a harmonic function as well. So we also can compress the wave-fronts and wave-normal rays into another analytical function

$$G(z) = z - \frac{a^2}{z}. \quad (55)$$

The real part represents the family of wave fronts, and the imaginary part represents the family of wave-normal rays exhibited in Fig. 5(b).

Owing to the energy conservation, the time-averaged Poynting vector $\langle \vec{S}_{(P)}^c \rangle$ is divergence-free. And the wave vector must be irrotational, since $\vec{k}_{(P)} = \nabla \psi$. However, if and only if the transformation satisfies Eq. (53), we can prove both $\langle \vec{S}_{(P)}^c \rangle$ and $\vec{k}_{(P)}$ are not only divergence-free but also irrotational. This conclusion is consistent to the fact that both light-rays and wave-normal rays satisfy Laplace's equation for the particular transformation function.

VI. CONCLUSION

In conclusion, we have meticulously compared the behaviors of electromagnetic fields in \mathbf{P} space with in \mathbf{S} system of virtual space, and provided the corresponding relations with

electromagnetic quantities and boundary conditions between the two systems. Then, we obtain the analytical expression of light-rays and wave-normal rays, and give their physical interpretation. Furthermore, we verify the expression of light-rays we obtained is exactly a particular solution of geodesic equation in \mathbf{S} system. In addition, using duality principle, we show that the covariant and contravariant components of wave vector in \mathbf{S} system map to wave vector and ray vector respectively in \mathbf{P} space, and it causes the split of $k_{(\mathbf{P})i}$ and $s_{(\mathbf{P})}^i$ in \mathbf{P} space. Finally, we find out the general form of transformation function which turns the light-ray to harmonic, and therefore we can construct a special cloak in which $\langle \vec{S}_{(\mathbf{P})}^c \rangle$ and $\vec{k}_{(\mathbf{P})}$ are both divergence-free and curl-free.

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APPENDIX A: DIFFERENT WAY TO DEFINE ELECTROMAGNETIC FIELDS IN CURVED SPACETIME

In Minkowski spacetime, 3-D macroscopic Maxwell's equations in a Lorentz frame are

$$\partial_i B^i = 0, \quad \frac{\partial B^i}{\partial t} + e^{ijk} \partial_j E_k = 0, \quad (\text{A.1a})$$

$$\partial_i D^i = \rho, \quad -\frac{\partial D^i}{\partial t} + e^{ijk} \partial_j H_k = 0, \quad (\text{A.1b})$$

and their 4-D covariant forms are

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0, \quad (\text{A.2a})$$

$$\partial_\alpha G^{\alpha\beta} = J^\beta. \quad (\text{A.2b})$$

The four 3-D electric and magnetic vectors, E_i , H_i , D^i and B^i are defined as

$$E_i = F_{0i}, \quad B^i = -\frac{1}{2c} e^{ijk} F_{jk}, \quad D^i = -\frac{1}{c} G^{0i}, \quad H_i = -\frac{1}{2} e_{ijk} G^{jk}. \quad (\text{A.3})$$

Here we use Greek indices α, β, \dots to run from 0 to 3 for time and spatial parts, and stipulate the sign of metric takes the form $(-, +, +, +)$. For vacuum, $G^{\alpha\beta} = c\varepsilon_0 F^{\alpha\beta}$.

In curved space time, 4-D microscopic Maxwell equations which have the same form as the macroscopic Maxwell's equations in vacuum become[21, 25]

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0, \quad (\text{A.4a})$$

$$c\varepsilon_0 \nabla_\alpha F^{\alpha\beta} = \frac{c\varepsilon_0}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} F^{\alpha\beta}) = J^\beta, \quad (\text{A.4b})$$

where ∇_α is 4-D covariant derivative, g is the determinant of the spacetime metric $g_{\mu\nu}$.

If the excitation tensor $G^{\alpha\beta}$ is defined in the form [4, 7, 26, 27]

$$G^{\alpha\beta} = c\varepsilon_0 \sqrt{-g} F^{\alpha\beta}, \quad (\text{A.5})$$

and the definitions of 3-D electric and magnetic vectors are retained for flat Minkowski spacetime as in Eqs. (A.3), both 3-D and 4-D Maxwell's equations keep the form of flat spacetime in Eqs. (A.1) and Eqs. (A.2), except the 4-current J^β , charge density ρ , and 3-current j^i should be replaced with the reduced forms $\hat{J}^\beta = \sqrt{-g} J^\beta$, $\hat{\rho} = \sqrt{\gamma} \rho$, $\hat{j}^i = \sqrt{\gamma} j^i$ respectively [7], where γ denotes the determinate of spacial metric $\gamma_{ij} = g_{ij} - g_{0i}g_{0j}/g_{00}$. We name this set of excitation tensor and 3-D electric and magnetic vectors as Minkowski's definition.

The constitutive relations in Minkowski's definition are [5, 7, 27]

$$D^i = \varepsilon_0 \sqrt{\frac{\gamma}{-g_{00}}} \gamma^{ij} E_j + \frac{1}{cg_{00}} e^{ijk} g_{0j} H_k, \quad (\text{A.6a})$$

$$B^i = \mu_0 \sqrt{\frac{\gamma}{-g_{00}}} \gamma^{ij} H_j - \frac{1}{cg_{00}} e^{ijk} g_{0j} E_k, \quad (\text{A.6b})$$

Thus, the permittivity and permeability satisfy $\varepsilon^{ij} = \varepsilon_0 \sqrt{-\gamma/g_{00}} \gamma^{ij}$, $\mu^{ij} = \mu_0 \sqrt{-\gamma/g_{00}} \gamma^{ij}$, and magnetoelectric coupling tensor is $\kappa^{ij} = e^{ijk} g_{0j}/(cg_{00})$. If $g_{\mu\nu}$ satisfies $g_{00} = -1$, $g_{0i} = 0$, then magnetoelectric coupling vanishes, while ε^{ij} and μ^{ij} reduce to Eqs. (4).

Another definition of 3-D electric and magnetic vectors appears in Landau's book [25], therefore, we call this set as Landau's definitions. In Landau's definition, we put a bar on each quantity, and the excitation tensor retains its formula in flat spacetime $\bar{G}^{\alpha\beta} = c\varepsilon_0 F^{\alpha\beta}$, and the four electric and magnetic vectors are redefined as [25, 28]

$$\bar{E}_i = F_{0i}, \quad \bar{B}^i = -\frac{1}{2c\sqrt{\gamma}} e^{ijk} F_{jk}, \quad \bar{D}^i = -\frac{\sqrt{-g_{00}}}{c} \bar{G}^{0i}, \quad \bar{H}_i = -\frac{\sqrt{-g}}{2} e_{ijk} \bar{G}^{jk}. \quad (\text{A.7})$$

Under this set of definitions, 4-D Maxwell's equations have the form of

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0, \quad (\text{A.8a})$$

$$\nabla_\alpha \bar{G}^{\alpha\beta} = J^\beta. \quad (\text{A.8b})$$

The corresponding 3-D Maxwell's equations are

$$\frac{1}{\sqrt{\gamma}}\partial_i(\sqrt{\gamma}\bar{B}^i) = 0, \quad \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}(\sqrt{\gamma}\bar{B}^i) + \frac{1}{\sqrt{\gamma}}e^{ijk}\partial_j\bar{E}_k = 0, \quad (\text{A.9a})$$

$$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^i}(\sqrt{\gamma}\bar{D}^i) = \rho, \quad -\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}(\sqrt{\gamma}\bar{D}^i) + \frac{1}{\sqrt{\gamma}}e^{ijk}\partial_j\bar{H}_k = J^i. \quad (\text{A.9b})$$

Comparing Eqs. (A.3) with Eqs. (A.7), we have $G^{\alpha\beta} = \sqrt{-g}\bar{G}^{\alpha\beta}$, and the transformations of other quantities between the two definitions are identical with Eqs. (2).

Assume that an observer whose 4-velocity is u^α in a curved spacetime endowed with an electromagnetic field $F_{\mu\nu}$. Then, questions arouse naturally. What is the measurement of electric and magnetic fields by the observer? Is the measurement equal to one of the two definitions mentioned above? Actually, there has been a mature way to answer this question. In the curved spacetime, electric and magnetic field vectors are defined respected with the frame u^α as [29]

$$\tilde{E}_\alpha = -F_{\alpha\beta}u^\beta, \quad \tilde{H}_\alpha = {}^*\bar{G}_{\alpha\beta}u^\beta = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\bar{G}^{\mu\nu}u^\beta, \quad (\text{A.10})$$

where ${}^*\bar{G}_{\alpha\beta} = \epsilon_{\alpha\beta\mu\nu}\bar{G}^{\mu\nu}/2$ is the dual of $\bar{G}^{\mu\nu}$. Here, we put a tilde on electric and magnetic vectors to discriminate its representation under Minkowski's definition and Landau's definition, however, the definition of excitation tensor retains its formula from Landau's definition $\bar{G}^{\alpha\beta} = c\epsilon_0 F^{\alpha\beta}$, which guarantees that \bar{G} is a tensor of 4-dimension.

In terms of the definitions of Eqs. (A.10), $\tilde{E}_\alpha u^\alpha = \tilde{H}_\alpha u^\alpha = 0$, which evinces that \tilde{E} and \tilde{H} are both space-like vectors for the observer u^α . In addition, if we simply let electric displacement $\tilde{D}^\alpha = \epsilon_0 \tilde{E}^\alpha$ and magnetic induction $\tilde{B}^\alpha = \mu_0 \tilde{H}^\alpha$, *i.e.* what are made in vacuum of flat spacetime, the components of \tilde{E} , \tilde{H} , \tilde{D} , and \tilde{B} in the tetrad $e_{(\mu)}^\alpha$ of the observer, where $e_{(0)}^\alpha = u^\alpha$, are

$$\tilde{E}_{\langle i \rangle} = F_{\langle 0i \rangle}, \quad \tilde{H}_{\langle i \rangle} = -\frac{1}{2}e_{ijk}\bar{G}^{\langle jk \rangle}, \quad \tilde{D}^{\langle i \rangle} = -\frac{1}{c}\bar{G}_{\langle 0i \rangle}, \quad \tilde{B}^{\langle i \rangle} = -\frac{c}{2}e^{ijk}F_{\langle jk \rangle}. \quad (\text{A.11})$$

As we see, the components in tetrad satisfy the original relation of flat spacetime (A.3). Furthermore, the electromagnetic tensors admit the following decomposition [29–31]:

$$F_{\mu\nu} = (\tilde{E}_\mu u_\nu - \tilde{E}_\nu u_\mu) - c\epsilon_{\mu\nu\alpha\beta}u^\alpha\tilde{B}^\beta, \quad (\text{A.12a})$$

$$\bar{G}^{\mu\nu} = c(\tilde{D}^\mu u^\nu - \tilde{D}^\nu u^\mu) - \epsilon^{\mu\nu\alpha\beta}u_\alpha\tilde{H}_\beta, \quad (\text{A.12b})$$

and a further study reveals the form of Maxwell's equations expressed with \tilde{E} and \tilde{H} [30, 31].

Here, we are interested in the condition that the observer is comoving. For a comoving observer, $u^\alpha = (1/\sqrt{-g_{00}}, 0, 0, 0)$, and the transverse projector $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ reduces to

the spatial metric γ_{ij} mentioned earlier. In this case, the relations (A.12) lead to the result

$$\left\{ \begin{array}{l} F_{0i} = \sqrt{-g_{00}} \tilde{E}_i, \\ F_{jk} = -c\sqrt{\gamma} e_{ijk} \tilde{B}^i + \frac{2}{\sqrt{-g_{00}}} \tilde{E}_{[j} g_{k]0}, \\ \bar{G}^{0i} = -\frac{c}{\sqrt{-g_{00}}} \tilde{D}^i + \frac{1}{\sqrt{g_{00}g}} e^{ijk} g_{j0} \tilde{H}_k, \\ \bar{G}^{jk} = -\frac{1}{\sqrt{\gamma}} e^{ijk} \tilde{H}_i. \end{array} \right. \quad (\text{A.13})$$

It seems that the measurement of electric and magnetic fields by an observer conforms neither to Minkowski's definition nor to Landau's definition. However, if we set $g_{00} = -1, g_{0i} = 0$, the reduced relations Eqs. (A.13) are identical with the reduced relations Eqs. (A.7) under Landau's definition. So, if we use artificial materials to simulate the behavior of electromagnetic fields in curved spacetime based on the form equivalence of Maxwell's equations, it should be heeded that the distinction between the electric and magnetic fields measured in materials and measured by the observer moving in real curved spacetime.

APPENDIX B: ELECTROMAGNETIC FIELDS IN CYLINDRICAL CLOAK

In this portion, we would only consider TE polarized wave. Substituting the constitutive parameters Eq. (8) into wave equation (37), we get the general scalar wave equation for all transformation $f(r)$ which is exactly the Helmholtz equation in original \mathbf{S}' coordinate system of virtual space:

$$\frac{1}{f} \frac{\partial}{\partial f} \left(f \frac{\partial E_{(\text{P})z}}{\partial f} \right) + \frac{1}{f^2} \frac{\partial^2 E_{(\text{P})z}}{\partial \theta^2} + k^2 E_{(\text{P})z} = 0. \quad (\text{B.1})$$

Through separating variables, electromagnetic fields can be represented by series

$$\vec{E}_{(\text{P})} = \sum_{-\infty}^{\infty} [a_n J_n(x) + b_n N_n(x)] e^{in\theta} \hat{e}_z, \quad (\text{B.2})$$

where $J_n(x)$ and $N_n(x)$ are n-order Bessel function and n-order Neumann function respectively. These expressions are available in every area, however x takes different forms in each area. Outside the cloak ($r > b$), $x = k_0 r$ is in the expressions of incident wave $\vec{E}_{(\text{P})}^{\text{in}}$ and scattering wave $\vec{E}_{(\text{P})}^{\text{sc}}$. In the cloak layer ($a < r < b$), $\vec{E}_{(\text{P})}^{\text{c}}$ takes $x = k_0 f(r)$. Inside the internal hidden area ($r < a$), $x = k_1 r$ is in the expression of $\vec{E}_{(\text{P})}^{\text{int}}$, where $k_1 = \omega \sqrt{\varepsilon_1 \mu_1}$ and ε_1, μ_1 are the material properties of the hidden medium. The incident wave $\vec{E}_{(\text{P})}^{\text{in}} = E_0 e^{ik_0 r \cos \theta} \hat{e}_z$ can be expanded as

$$\vec{E}_{(\text{P})}^{\text{in}} = E_0 \sum_{-\infty}^{\infty} i^n J_n(k_0 r) e^{in\theta}. \quad (\text{B.3})$$

Because \vec{E}^{sc} should tend to an outgoing wave as r approaches to infinity, it is more convenient to expand scattering wave with Hankel function of first kind:

$$\vec{E}_{(\text{P})}^{\text{sc}} = \sum_{-\infty}^{\infty} a_n^{\text{sc}} H_n^{(1)}(k_0 r) e^{in\theta}. \quad (\text{B.4})$$

In addition, $b_n^{\text{int}} = 0$ means that fields should be finite when $r \rightarrow 0$. Then applying the continuity of E_z and H_θ at inner and outer surface of cloak, we have following equations:

$$a_n^{\text{int}} J_n(k_1 a) = a_n^c J_n(k_0 f(a)) + b_n^c N_n(k_0 f(a)), \quad (\text{B.5a})$$

$$a_n^{\text{int}} \frac{\mu_0}{\mu_1} k_1 J_n(k_1 a) = \frac{k_0 f(a)}{a} [a_n^c J_n'(k_0 f(a)) + b_n^c N_n'(k_0 f(a))], \quad (\text{B.5b})$$

$$E_0 i^n J_n(k_0 b) + a_n^{\text{sc}} H_n^{(1)}(k_0 b) = a_n^c J_n(k_0 f(b)) + b_n^c N_n(k_0 f(b)), \quad (\text{B.5c})$$

$$E_0 i^n J_n'(k_0 b) + a_n^{\text{sc}} H_n^{(1)'}(k_0 b) = \frac{f(b)}{b} [a_n^c J_n'(k_0 f(b)) - b_n^c N_n'(k_0 f(b))], \quad (\text{B.5d})$$

Solving the group of equations, we obtain

$$a_n^c = E_0 i^n [J_n(k_0 b) H_n^{(1)'}(k_0 b) - J_n'(k_0 b) H_n^{(1)}(k_0 b)] / \left\{ H_n^{(1)'}(k_0 b) [J_n(k_0 f(b)) + A_n N_n(k_0 f(b))] - \frac{f(b)}{b} H_n^{(1)}(k_0 b) [J_n'(k_0 f(b)) + A_n N_n'(k_0 f(b))] \right\}, \quad (\text{B.6a})$$

$$b_n^c = \frac{\mu_0 a k_1 J_n(k_0 f(a)) J_n'(k_1 a) - \mu_1 k_0 f(a) J_n'(k_0 f(a)) J_n(k_1 a)}{\mu_1 k_0 f(a) N_n'(k_0 f(a)) J_n(k_1 a) - \mu_0 a k_1 N_n(k_0 f(a)) J_n'(k_1 a)} a_n^c = A_n a_n^c, \quad (\text{B.6b})$$

$$a_n^{\text{int}} = \frac{1}{J_n(k_1 a)} [J_n(k_0 f(a)) + A_n N_n(k_0 f(a))] a_n^c, \quad (\text{B.6c})$$

$$a_n^{\text{sc}} = \frac{1}{H_n^{(1)'}(k_0 b)} \left\{ \frac{f(b)}{b} [J_n'(k_0 f(b)) - A_n N_n'(k_0 f(b))] a_n^c - E_0 i^n J_n'(k_0 b) \right\}. \quad (\text{B.6d})$$

Substituting invisibility condition $f(a) = 0$, $f(b) = b$, we have $a_n^c = E_0 i^n$, $b_n^c = 0$ (*i.e.* $A_n = 0$), therefore $a_n^{\text{sc}} = 0$ which indicates no scattering. However, when calculating a_n^{int} , there is a term $A_n \cdot N_n(k_0 f(a))$, in which $A_n \rightarrow 0$ while $N_n(k_0 f(a)) \rightarrow \infty$ as $f(a) \rightarrow 0$. A meticulous calculation gives the result $a_n^{\text{int}} \rightarrow 0$, so waves can not spread into the hidden area. Finally, we get the expressions of electromagnetic fields in the whole space as shown in Eqs.(24).

A detailed derivation should be identified. If we substitute the invisibility condition $f(a) = 0$, $f(b) = b$ at the beginning, the fields in the cloak layer must have no terms of Neumann function, which means $b_n^c = 0$, since $N_n(k_0 f(a)) \rightarrow \infty$ is against the requirement that fields should be finite. Therefore, the last terms vanish in Eqs. (B.5a) and (B.5b). In this case, the Eq. (B.5b) still leads to $a_n^{\text{int}} = 0$, yet the result can not satisfy Eq. (B.5a) when $n = 0$. In the more general derivation above, when $n = 0$, though $b_n^c \rightarrow 0$, the product $b_n^c \cdot N_n(k_0 f(a))$ is towards to a finite quantity $-E_0 i^n / J_n(k_1 a)$, and it counterbalances the equality (B.5a), however, the product only exists at the inner surface, so it no longer has

the meaning of fields but represents the magnetic surface current flowing along the z axis [13].

APPENDIX C: EIKONAL EQUATION OF DIFFERENT FORMS

Wave equation (37) can be written as the components form

$$\left[\epsilon_{ipj} k^p (\bar{n}^{-1})^{jk} \epsilon_{kql} k^q + \left(\frac{\omega}{c} \right)^2 \bar{n}_{il} \right] E^l = 0. \quad (C.1)$$

The condition which protects E^l from trivial solution is

$$\det \left[\epsilon_{ipj} k^p (\bar{n}^{-1})^{jk} \epsilon_{kql} k^q + \left(\frac{\omega}{c} \right)^2 \bar{n}_{il} \right] = 0. \quad (C.2)$$

Expanding the determinant

$$\begin{aligned} & \det \left[\epsilon_{ipj} k^p (\bar{n}^{-1})^{jk} \epsilon_{kql} k^q + \left(\frac{\omega}{c} \right)^2 \bar{n}_{il} \right] \\ &= e^{i_1 i_2 i_3} \cdot \prod_{\alpha=1}^3 \left[\epsilon_{i_\alpha p_\alpha j_\alpha} k^{p_\alpha} (\bar{n}^{-1})^{j_\alpha k_\alpha} \epsilon_{k_\alpha q_\alpha \alpha} k^{q_\alpha} + \left(\frac{\omega}{c} \right)^2 \bar{n}_{i_\alpha \alpha} \right] \\ &= \sqrt{\gamma} \epsilon^{i_1 i_2 i_3} \left[\prod_{\alpha=1}^3 M_{i_\alpha \alpha} + \left(\frac{\omega}{c} \right)^2 \sum_{\alpha=1}^3 \bar{n}_{i_\alpha \alpha} M_{i_\alpha(\alpha+1)} M_{i_\alpha(\alpha+2)} \right. \\ & \quad \left. + \left(\frac{\omega}{c} \right)^4 \sum_{\alpha=1}^3 M_{i_\alpha \alpha} \bar{n}_{i_{\alpha+1}(\alpha+1)} \bar{n}_{i_{\alpha+2}(\alpha+2)} + \left(\frac{\omega}{c} \right)^6 \prod_{\alpha=1}^3 \bar{n}_{i_\alpha \alpha} \right] \end{aligned} \quad (C.3)$$

where $M_{il} = \epsilon_{ipj} N_l^{pj} = \epsilon_{ipj} k^p (\bar{n}^{-1})^{jk} \epsilon_{kql} k^q$, and the indices of $\epsilon_{i_\alpha j_\alpha k_\alpha}$ satisfy $\alpha \bmod 3$. Calculating each term in the expansion, we have

$$\begin{aligned} \text{1st term:} \quad & \sqrt{\gamma} \epsilon^{i_1 i_2 i_3} \prod_{\alpha=1}^3 M_{i_\alpha \alpha} = \sqrt{\gamma} \epsilon^{i_1 i_2 i_3} \epsilon_{i_1 p_1 j_1} \epsilon_{i_2 p_2 j_2} \epsilon_{i_3 p_3 j_3} N_1^{p_1 j_1} N_2^{p_2 j_2} N_3^{p_3 j_3} \\ &= \sqrt{\gamma} (\epsilon_{p_1 p_2 j_2} \epsilon_{j_1 p_3 j_3} - \epsilon_{j_1 p_2 j_2} \epsilon_{p_1 p_3 j_3}) N_1^{p_1 j_1} N_2^{p_2 j_2} N_3^{p_3 j_3} = 0, \end{aligned} \quad (C.4)$$

$$\begin{aligned} \text{2nd term:} \quad & \sqrt{\gamma} \left(\frac{\omega}{c} \right)^2 \sum_{\alpha=1}^3 \epsilon^{i_1 i_2 i_3} \bar{n}_{i_\alpha \alpha} M_{i_\alpha(\alpha+1)} M_{i_\alpha(\alpha+2)} \\ &= \sqrt{\gamma} \left(\frac{\omega}{c} \right)^2 \sum_{\alpha=1}^3 \epsilon^{i_\alpha i_{\alpha+1} i_{\alpha+2}} \bar{n}_{i_\alpha \alpha} M_{i_\alpha(\alpha+1)} M_{i_\alpha(\alpha+2)} \\ &= \sqrt{\gamma} \left(\frac{\omega}{c} \right)^2 \sum_{\alpha=1}^3 \bar{n}_{i_\alpha \alpha} \epsilon_{p_{\alpha+2} j_{\alpha+1} j_{\alpha+2}} N_{\alpha+1}^{p_{\alpha+1} j_{\alpha+1}} N_{\alpha+2}^{p_{\alpha+2} j_{\alpha+2}}, \end{aligned}$$

because

$$\begin{aligned}
& \text{2nd term} \cdot \det(\bar{n}_{ij}) \\
&= \sqrt{\gamma} \left(\frac{\omega}{c}\right)^2 \sum_{\alpha=1}^3 \bar{n}_{i\alpha} \epsilon_{p\alpha+2j\alpha+1j\alpha+2} \cdot \left[k^{i\alpha} (\bar{n}^{-1})^{j\alpha+1k\alpha+1} \epsilon_{k\alpha+1q\alpha+1(\alpha+1)} k^{q\alpha+1} \right] \\
&\quad \cdot \left[k^{p\alpha+2} (\bar{n}^{-1})^{j\alpha+2k\alpha+2} \epsilon_{k\alpha+2q\alpha+2(\alpha+2)} k^{q\alpha+2} \right] \cdot \left[\sqrt{\gamma} \epsilon^{t_1 t_2 t_3} \bar{n}_{t_1 1} \bar{n}_{t_2 2} \bar{n}_{t_3 3} \right] \\
&= \gamma \left(\frac{\omega}{c}\right)^2 \left[\bar{n}_{i\alpha} \bar{n}_{t\beta} k^{i\alpha} k^{t\beta} \gamma k^\alpha k^\beta \right] = \left(\gamma \frac{\omega}{c} \bar{n}_{ij} k^i k^j \right)^2,
\end{aligned}$$

therefore we have

$$\text{2nd term} = \left(\gamma \frac{\omega}{c} \bar{n}_{ij} k^i k^j \right)^2 / \det(\bar{n}_{ij}), \quad (\text{C.5})$$

$$\begin{aligned}
\text{3rd term:} \quad & \sqrt{\gamma} \left(\frac{\omega}{c}\right)^4 \sum_{\alpha=1}^3 \epsilon^{i_1 i_2 i_3} M_{i\alpha} \bar{n}_{i\alpha+1(\alpha+1)} \bar{n}_{i\alpha+2(\alpha+2)} \\
&= \sqrt{\gamma} \left(\frac{\omega}{c}\right)^4 \sum_{\alpha=1}^3 \epsilon^{i\alpha i\alpha+1 i\alpha+2} \epsilon_{i\alpha p\alpha j\alpha} \bar{n}_{i\alpha+1(\alpha+1)} \bar{n}_{i\alpha+2(\alpha+2)} N_\alpha^{p\alpha j\alpha} \\
&= \sqrt{\gamma} \left(\frac{\omega}{c}\right)^4 \sum_{\alpha=1}^3 \left[\bar{n}_{p\alpha(\alpha+1)} \epsilon_{q\alpha\alpha(\alpha+2)} - \bar{n}_{p\alpha(\alpha+2)} \epsilon_{q\alpha\alpha(\alpha+1)} \right] k^{p\alpha} k^{q\alpha} \\
&= -2\gamma \left(\frac{\omega}{c}\right)^4 \bar{n}_{ij} k^i k^j, \quad (\text{C.6})
\end{aligned}$$

$$\text{4th term:} \quad \sqrt{\gamma} \left(\frac{\omega}{c}\right)^6 \epsilon^{i_1 i_2 i_3} \prod_{\alpha=1}^3 \bar{n}_{i\alpha} = \left(\frac{\omega}{c}\right)^6 \det(\bar{n}_{ij}). \quad (\text{C.7})$$

Substituting them into Eq. (C.3), we have

$$\det \left[\epsilon_{ipj} k^p (\bar{n}^{-1})^{jk} \epsilon_{kql} k^q + \left(\frac{\omega}{c}\right)^2 \bar{n}_{il} \right] = \frac{(\gamma\omega/c)^2}{\det(\bar{n}_{ij})} \left[\bar{n}_{ij} k^i k^j - \frac{1}{\gamma} \left(\frac{\omega}{c}\right)^2 \det(\bar{n}_{ij}) \right]^2 = 0. \quad (\text{C.8})$$

This is exactly the wave-vector eikonal equation (38) expressed by Landau's definition. The ray-vector eikonal equation (40) can be simply obtained through duality rules. In terms of Eq. (2b) and Eq. (5), we can get the two kinds of eikonal equation written by Minkowski's components and anholonomic components.

Under Minkowski's definition, wave-vector eikonal equation is

$$n^{ij} k_i k_j - \left(\frac{\omega}{c}\right)^2 \det(n^{ij}) = n_{ij} k^i k^j - \frac{1}{\gamma^2} \left(\frac{\omega}{c}\right)^2 \det(n_{ij}) = 0, \quad (\text{C.9})$$

and ray-vector eikonal equation is

$$(n^{-1})^{ij} s_i s_j - \gamma^2 \left(\frac{c}{\omega}\right)^2 \det[(n^{-1})^{ij}] = (n^{-1})_{ij} s^i s^j - \left(\frac{c}{\omega}\right)^2 \det[(n^{-1})_{ij}] = 0. \quad (\text{C.10})$$

The wave-vector eikonal equation expressed by the components in anholonomic unit basis is

$$(n^{-1})_{\langle ij \rangle} k_{\langle i \rangle} k_{\langle j \rangle} - \left(\frac{\omega}{c} \right)^2 \det [(n^{-1})_{\langle ij \rangle}] = 0, \quad (\text{C.11})$$

and the ray-vector eikonal equation is

$$(n^{-1})_{\langle ij \rangle} s_{\langle i \rangle} s_{\langle j \rangle} - \left(\frac{c}{\omega} \right)^2 \det [(n^{-1})_{\langle ij \rangle}] = 0. \quad (\text{C.12})$$

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